①

p-adic parallel transport, following Deninger–Werner, Würthen

Recall: ~~[crossed out]~~ X = compact Kähler manifold, then there is a 1-1 bijection between
- irreducible numerically flat vector bundles on X
- irreducible unitary representations $\pi_1(X) \longrightarrow U(r)$.

Defn: ~~numeri~~ $\mathcal{E}/X$ is numerically flat if
(1) both $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1)$ are numerically effective.
(2) (when X is projective) $\forall f: C \longrightarrow X$ from a smooth proj curve C, $f^*\mathcal{E}$ is semistable of degree 0.

Functor given by $\rho \rightsquigarrow \mathbb{L}_\rho \underset{\underline{\mathbb{C}}_X}{\otimes} \mathcal{O}_X$.

<u>Today:</u> try to extend this to p-adic setting.
~~plan~~ 1. motivate and explain certain ~~notion~~ concepts.
2. state the results.
3. indicate proof.

1. $\pi_1(X,x)$ in topological may be thought of as compatible Deck transformation of all the covering spaces of X:

$$\begin{array}{c} Y \\ \text{covering space} \downarrow f \\ x \in X \end{array} \quad \text{then any } g \in \pi_1(X,x) \text{ determines } \mathrm{Aut}(f^{-1}\{x\}).$$
$$\rightsquigarrow \pi_1(X,x) \longrightarrow \mathrm{Aut}(f^{-1}\{x\})$$

In ~~alge~~ AG, we may talk about finite covering spaces of X
= Category of finite étale schemes over $X = X_{fét}$.

choose $\bar{x} \longrightarrow X$ a geometric pt. We get functor
$$X_{fét} \xrightarrow{F_{\bar{x}}} \mathrm{FSets}$$
$$Y \longmapsto Y_{\bar{x}}$$

Similarly, ~~we~~ Grothendieck defined:

Defn $\pi_1(X, \bar{x}) = \mathrm{Aut}(F_{\bar{x}})$.

Properties:

~~Then Sin ...~~ • We get correspondence:

$\{\pi_1(X, \bar{x}) \longrightarrow G = \text{finite gp}\}/\sim \longleftrightarrow \{\text{finite étale } G\text{-cover of } X\}/\sim$.

- $\pi_1(X, \bar{x})$ doesn't depend on the choice of $\bar{x}$.
- $\pi_1(X, \bar{x})$ is naturally a profinite gp.
- for $X/\mathbb{C}$, we have $\pi_1(X, \bar{x}) \cong \pi_1^{top}(X(\mathbb{C}))^\wedge$ (profinite cpltion).

Now let's assume $X/\mathbb{C}_p$ smooth proper (rigid) variety.

So we have one side object: representations

$$\pi_1(X, \bar{x}) \longrightarrow GL_n(\mathcal{O}_{\mathbb{C}_p}) = \lim_n GL_n(\mathbb{Z}_p/p^n)$$

which corresponds to $\mathcal{O}_{\mathbb{C}_p}$-local ~~pro-finite~~ system $\mathbb{L}$.

$$= \lim_n \operatorname*{colim}_{\substack{\mathbb{Q}_p \subseteq K \subseteq \overline{\mathbb{Q}}_p \\ \text{finite}}} GL_n(\mathcal{O}_K/p^n).$$

We immediately see 2 problems (inter-related):

- X_{et} is <u>not</u> local acyclic
- $\mathbb{L}$ is <u>not</u> a sheaf on X_{et}. $(T_p \mathcal{H}^1(\mathbb{Z}_p) \neq 0)$.

So we <u>cannot</u> make sense of "$\mathbb{L} \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{O}_X$".

Solution: pro-étale site!

one enlarges the objects of covering allowing towers of étale covering, so that one automatically gets local acyclicity and any p-adic local system is naturally a sheaf on X_{proet}.

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Illustrating example:

$$\left(\cdots \longrightarrow \mathbb{G}_m^r \xrightarrow{(-)^p} \mathbb{G}_m^r \xrightarrow{(-)^p} \mathbb{G}_m^r\right) =: \widetilde{\mathbb{G}}_m^r \longrightarrow \mathbb{G}_m^r$$

is a new kind of covering allowed in $(\mathbb{G}_m^r)_{\text{pro-ét}}$.

Heuristic: the π_1 of n-th level $\mathbb{G}_m^r$ may be regarded as $p^n \cdot \pi_1$ of base $\mathbb{G}_m^r$, so intuitively the "limit" will have *really small* π_1, hence probably having local acyclicity.

In p-adic geometry, we have ~~several~~ interesting structure sheaves on $X_{\text{pro-ét}}$.

$$\mathcal{O}_X^+ := \left(U = \lim U_i \longrightarrow X\right) \longmapsto \operatorname{colim}_i \left\{\begin{array}{l}\text{functions on } U_i \text{ with} \\ \text{p-adic norm } \leq 1\end{array}\right\}.$$

$\mathcal{O}_X := \mathcal{O}_X^+[\frac{1}{p}]$.

$\widehat{\mathcal{O}}_X^+ := \lim_n \mathcal{O}_X^+/p^n \qquad \widehat{\mathcal{O}}_X := \widehat{\mathcal{O}}_X^+[\frac{1}{p}]$.

on $\widetilde{\mathbb{G}}_m^r \longrightarrow \mathbb{G}_m^r \left(= \mathrm{Sp}(\mathbb{C}_p\langle T_i^{\pm 1}\rangle)\right)$ (where now $\mathbb{G}_m^r$ is the rigid-analytic torus $\mathrm{Sp}(\mathbb{C}_p\langle T_i^{\pm 1}\rangle)$)

we have

$$\mathcal{O}_X^+(\widetilde{\mathbb{G}}_m^r) = \operatorname{colim}_n \mathcal{O}_{\mathbb{C}_p}\langle T_i^{\pm 1/p^n}\rangle.$$

$$\mathcal{O}_X(\widetilde{\mathbb{G}}_m^r) = \operatorname{colim}_n \mathbb{C}_p\langle T_i^{\pm 1/p^n}\rangle.$$

$$\widehat{\mathcal{O}}_X^+(\widetilde{\mathbb{G}}_m^r) \approx \mathcal{O}_{\mathbb{C}_p}\langle T_i^{\pm 1/p^\infty}\rangle.$$

$$\widehat{\mathcal{O}}_X(\widetilde{\mathbb{G}}_m^r) = \mathbb{C}_p\langle T_i^{\pm 1/p^\infty}\rangle.$$

§2. Thms / statements / constructions

Thm 1 / construction: Assume $\mathcal{E}^+$ is a finite rk locally free $\mathcal{O}_X^+$-module on X such that $\exists$ profinite étale cover $\widetilde{X} = \lim_i X_i \longrightarrow X$ such that

④ $\widehat{\mathcal{E}^+}\big|_{\widetilde{X}} \cong \left(\mathcal{E}^+ \underset{\mathcal{O}_X^+}{\otimes} \widehat{\mathcal{O}_X^+}\right)\Big|_{\widetilde{X}}$ is trivial $\left(\simeq \left(\widehat{\mathcal{O}_{\widetilde{X}}^+}\right)^{\oplus r}\right)$.

Then there exists a unique (up to isom) representation:

$$\rho_{\mathcal{E}^+} : \pi_1(X) \longrightarrow GL_r(\mathcal{O}_{\mathbb{C}_p}),$$

corresponding to a local system $\mathbb{L}$ such that one have a natural isom:

$$\widehat{\mathcal{E}^+} \cong \mathbb{L} \underset{\mathcal{O}_{\mathbb{C}_p}}{\otimes} \widehat{\mathcal{O}_X^+}.$$

Moreover, this association is compatible with all kinds of linear-algebraic operation you can think of, and it doesn't depend on the choice of $\widetilde{X}$.

Defn: $VB^{pfé}(\mathcal{O}_X^+)$ denotes the category of those $\mathcal{E}^+$'s show up in previous thm.

$$VB^{pfé}(\mathcal{O}_X) := VB^{pfé}(\mathcal{O}_X^+) \underset{\mathcal{O}_{\mathbb{C}_p}}{\otimes} \mathbb{C}_p = VB^{pfé}(\mathcal{O}_X^+)[\tfrac{1}{p}].$$

Thm 2: the previous thm gives a functor

$$\rho : VB^{pfé}(\mathcal{O}_X) \longrightarrow Rep_{\pi_1(X)}(\mathbb{C}_p)$$

which is fully faithful, and again we have

$$\mathcal{E} \underset{\mathcal{O}_X}{\otimes} \widehat{\mathcal{O}_X} \cong \mathbb{L}_{\rho(\mathcal{E})} \underset{\mathbb{C}_p}{\otimes} \widehat{\mathcal{O}_X}.$$

~~Intuition~~ Rmk: there are 2 ways of thinking $\mathcal{E}^+$:

① it's an integral model of a vector bundle:

$$\begin{tikzcd} \mathcal{E}/X \arrow[r, hook] \arrow[d] & \mathfrak{X} \backslash \mathcal{E}^+ \arrow[d] \\ \mathbb{C}_p \arrow[r, hookleftarrow] & \mathcal{O}_{\mathbb{C}_p} \end{tikzcd}$$

(algebraic point of view)

2 one may think of it as some kind of a metric on $\mathcal{E}$. (analytic point of view, which is ~~more closer~~ more analogous to Simpson's work over $\mathbb{C}$).

Q: How the hell can we verify that some vector bundle on X is in $VB^{pfé}(\mathcal{O}_X)$??

Thm 3: Suppose $\mathcal{E}/X$ is a vector bundle, such that $\exists$ $\begin{array}{c} X' \\ f \downarrow f.ét \\ X \end{array}$ and $f^*\mathcal{E}/X'$ has an integral model $\mathcal{F}/\mathfrak{X}'$ such that $\mathcal{F}|_{\overline{\mathbb{F}}_p}/\mathfrak{X}'_{\overline{\mathbb{F}}_p}$ is numerically flat.

$$\begin{array}{c} \mathfrak{X}' \\ \downarrow \\ \mathcal{O}_{\mathbb{C}_p} \end{array}$$

Then $\mathcal{E} \in VB^{pfé}(\mathcal{O}_X)$.

Using thm 3 and knowledge of picard varieties, one can show

Thm 4: $Pic^{\tau}_X(\mathbb{C}_p) \subseteq VB^{pfé}(\mathcal{O}_X)$.

§3. "Proof" of these.

"pf" of Thm 1: Step 1: we can make those X_i's showing up in the trivializing tower to be connected.

Step 2: under connectivity assumption: $\Gamma(\widetilde{X}, \widehat{\mathcal{O}}^+_X) = \mathcal{O}_{\mathbb{C}_p}$.

Hence $\Gamma(\widetilde{X}, \hat{\mathcal{E}}^+|_{\widetilde{X}}) \xleftarrow{\cong} \mathcal{O}_{\mathbb{C}_p}^{\oplus r}$.

Step 3/construction:
$$\begin{tikzcd} & \widetilde{X} \arrow[d] \\ \bar{x} \arrow[r] & X \end{tikzcd}$$
we have $\widetilde{X}_{\bar{x}} = \lim\limits_{i} (X_i)_{\bar{x}}$ (profinite set, hence non-empty!)

choose $\tilde{x} \to \widetilde{X}_{\bar{x}} \subseteq \widetilde{X}$ $\forall g \in \pi_1(X, \bar{x})$, $g \cdot \tilde{x}$ gives another point $g \cdot \tilde{x} \longrightarrow \widetilde{X}_{\bar{x}} \subseteq \widetilde{X}$.

$\tilde{x}^*: \Gamma(\tilde{X}, \hat{\mathcal{E}}^+) \xrightarrow{\cong} \Gamma(\bar{x}^* \hat{\mathcal{E}}^+)$, so we get

$\hat{\mathcal{E}}^+_{\bar{x}} \xrightarrow{(\tilde{x}^*)^{-1}} \Gamma(\tilde{X}, \hat{\mathcal{E}}^+) \xrightarrow{(g\cdot\tilde{x})^*} \hat{\mathcal{E}}^+_{\bar{x}}$. (note $\hat{\mathcal{E}}^+_{\bar{x}}$ is canonically identified with $\hat{\mathcal{E}}^+_{\tilde{x}}$)

step 4: show that this map from $\hat{\mathcal{E}}^+_{\bar{x}}$ to $\hat{\mathcal{E}}^+_{\bar{x}}$ doesn't depend on the choices of $\tilde{X}$ and $\tilde{x}$.

Step 5: hence we get $\pi_1(X, \bar{x}) \longrightarrow GL(\hat{\mathcal{E}}^+_{\bar{x}})$ homomorphism.

The rest needs to be argued from very definition... □

"pf" of Thm 3: Reduction: we may replace $(\mathcal{E}, X)$ by $(f^*\mathcal{E}, X')$.
and $\mathcal{F}$ gives $\mathcal{E}^+$. and $(\mathcal{F}_0, \mathcal{X}_0)$ defined over $\mathbb{F}_q$.

step 1: by definition $(\mathrm{Frob}^s)^* \mathcal{F}_0$ are all numerically flat on $\mathcal{X}_0$.

Fact: Langer showed that if $\mathcal{X}_0$ is proj., there are finitely many numerically flat bundles of rk r on $\mathcal{X}_0/\mathbb{F}_q$.

So we get that $(F^{s_1})^* \mathcal{F}_0 \simeq (F^{s_2})^* \mathcal{F}_0$. (in general, have to use Bhatt-Scholze's v-descent on X_{perf})

Fact: in this case, we may find $\mathcal{Z}_0 \xrightarrow{f.\acute{e}t} \mathcal{X}_0$ s.t. $(F^{s_1})^* \mathcal{F}_0 |_{\mathcal{Z}_0}$ is trivial.

Step 2: hence we get $Z \xrightarrow{f.\acute{e}t} X$ s.t. $\mathcal{E}^+/\pi |_Z$ is $(\mathcal{O}^+_X/p^{\frac{1}{p^\infty}} \xrightarrow{\mathrm{Frob}} \cong$ trivial.
(use some canonical spread out) some $\pi \in \mathfrak{m}_{\bar{\mathbb{Z}}_p}$.

Step 3: $0 \to \pi\mathcal{E}^+/\pi^2 \longrightarrow \mathcal{E}^+/\pi^2 \longrightarrow \mathcal{E}^+/\pi \to 0$ gives finitely many extension classes of $\mathrm{Ext}^1(\mathcal{O}^+_{X/\pi}, \mathcal{O}^+_{X/\pi}) \cong H^1(\mathcal{O}^+_{X/\pi})$ (replacing X w/ Z)
which by Scholze's primitive comparison is almost

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isomorphic to $H^1(X, \mathbb{F}_p) \otimes \mathcal{O}_{\mathbb{C}_p/\pi}$

- any class in $H^1(X, \mathbb{F}_p)$ may be killed by finite étale cover.
- $$\begin{tikzcd}
0 \arrow[r] & \pi\mathcal{E}^+/\pi^2 \arrow[r] & \mathcal{E}^+/\pi^2 \arrow[r] & \mathcal{E}^+/\pi \arrow[r] & 0 \\
0 \arrow[r] & \pi \cdot p^{-\varepsilon}\mathcal{E}^+/\pi^2\cdot p^{-\varepsilon} \arrow[r] \arrow[u, "\cdot p^{\varepsilon}"] & \mathcal{E}^+/\pi^2\cdot p^{-\varepsilon} \arrow[r] \arrow[u, "\cdot p^{\varepsilon}"] & \mathcal{E}^+/\pi\cdot p^{-\varepsilon} \arrow[r] \arrow[u, "\cdot p^{\varepsilon}"] & 0
\end{tikzcd}$$

so $\forall\, \varepsilon > 0$, we may trivialize after some f.ét cover.

Step 4: keep doing so, may trivialize

$$\mathcal{E}^+/\pi^{n-\sum_{i=1}^{n}\varepsilon_i}.$$

choosing ε_i cleverly, we may trivialize

$$\lim_n \mathcal{E}^+/\pi^{n-\sum_{i=1}^{n}\varepsilon_i} \cong \widehat{\mathcal{E}^+}$$

by a f.ét covering tower.

"pf" of Thm 4: • for any line bundle in $(\hat{\beta}^{\Delta} \cong)$ generic fiber of identity of Néron model of Pic^0_X, we may find numerically flat reduction directly.

- $\mathrm{Pic}^{\tau}_X(\mathbb{C}_p) \supseteq \hat{\beta}(\mathbb{C}_p)$ with cokernel being torsion.

so $\forall\, \mathcal{L} \in \mathrm{Pic}^{\tau}_X(\mathbb{C}_p)$, $\mathcal{L}^m$ has numerically flat reduction.

and we can find $\mathcal{L}' \in \hat{\beta}$ s.t. $\mathcal{L}'^m = \mathcal{L}^m$

(~~semi~~ $[m]$ is always a surjection on semi-abelian varieties)

so $\mathcal{L} \cong \mathcal{L}' \otimes$ m-torsion

← can be trivialized by a f.ét cover.